Workshop Report:

Recent advances on modeling multiple discrete-continuous choices

Abdul Pinjari
University of South Florida
Department of Civil and Environmental Engineering
Email: apinjari@usf.edu

Chandra Bhat
The University of Texas at Austin
Department of Civil, Architectural and Environmental Engineering
Email: bhat@mail.utexas.edu

David Bunch
University of California, Davis
Graduate School of Management
Email: dsbunch@ucdavis.edu

to appear in

Choice modelling: the state of the art and the state of practice: *Selected papers from the Second International Choice Modelling Conference.*

Edited by Stephane Hess and Andrew Daly.

Cite as: Pinjari, A.R., Bhat C.R., and D.S. Bunch (2013) Workshop Report: Recent Advances on Multiple Discrete-Continuous Choice Models. *Choice modelling: the state of the art and the state of practice*: *Selected papers from the Second International Choice Modelling Conference.* Edited by Stephane Hess and Andrew Daly.

Abstract

This report summarizes the workshop discussions on recent advances and future directions for modeling multiple discrete-continuous choices.

1. Introduction

Many consumer choice situations involve making a selection (or selections) from among a set of competing (discrete) alternatives, along with a set of corresponding (continuous) quantity decisions. An important, widely studied special case is the discrete choice of a single alternative (assuming all consumption is confined to the single chosen alternative). However, situations involving multiple discrete-continuous (MDC) choices are pervasive in the social sciences, including transportation, economics and marketing. Examples include individuals' time-use choices (decisions to engage in different types of activities and time allocation to each activity), investment portfolios (where and how much to invest), and grocery purchases (brand choice and purchase quantity).

A number of different approaches have been used in the literature to handle "multiple discreteness." One approach is to enumerate all possible bundles of the elemental choice alternatives and treat each bundle as a "composite" alternative within a traditional, single discrete choice framework. A problem with this approach is that the number of composite alternatives explodes as the number of elemental alternatives increases. A second approach is to use a multivariate statistical system, with several univariate discrete choice model equations linked to each other through statistical correlations. This reduced-form method is based on a rather mechanical statistical "stitching" of multiple univariate model equations rather than a unified, underlying theoretical framework. One specific approach that addresses both multiple discreteness

and quantity choice assumes that a decision made at a single point in time (e.g., a shopping trip) must address multiple consumption occasions over a succeeding time period (e.g., before the next shopping trip). Because taste for products as well as the number of consumption occasions can vary during the time period, an optimal purchase decision will include multiple alternatives in varying quantities (Hendel 1999, Dubé 2004). A more general approach, and the focus in this workshop, uses the classical microeconomic theory of constrained utility maximization as a starting point for deriving alternative models from specified sets of assumptions.

The rest of this paper is organized in four sections. The next section provides an overview of the classical microeconomic theory of constrained utility maximization as applied to MDC choices. Two theoretically equivalent yet distinct approaches are discussed: (1) the direct utility maximization approach (also called the Kuhn-Tucker approach), and (2) the indirect utility approach. Section 3 reviews recent advances in the context of the Kuhn-Tucker (KT) approach to model MDC choices. Section 4 identifies directions for further research. Section 5 concludes with a summary.

2. The Random Utility Maximization (RUM) approach to modeling MDC choices

Consumers are assumed to maximize a direct utility function $U(\mathbf{x})$ over a set of non-negative consumption quantities $\mathbf{x} = (x_1, ..., x_k, ..., x_K)$ subject to a linear budget constraint, as below:

$$\textit{Max } U(\mathbf{x}) \text{ such that } \mathbf{x}.\mathbf{p} = y \text{ and } x_k \geq 0 \qquad (1)$$

where $U(\mathbf{x})$ is a quasi-concave, increasing and continuously differentiable utility function with respect to the consumption quantity vector $\mathbf{x}$, $\mathbf{p}$ is the vector of unit prices for all

goods, *y* is a budget for total expenditure, and the consumer index in Equation (1) is suppressed for clarity of presentation.

The vector $\mathbf{x}$ in Equation (1) may or may not include an *outside good*. The outside good, when included, is a composite good that represents all goods other than the *K-1 inside* goods of interest to the analyst. Generally, the outside good is treated as a numeraire with unit price, implying that the prices and characteristics of all goods grouped into the outside category do not influence the choice and expenditure allocation among the inside goods (see Deaton and Muelbauer 1980). The outside good allows the overall demand for the inside goods to change due to changes in their prices and other influential factors. Classical microeconomic analysis typically considers the allocation of expenditures across broad categories (e.g., food, housing, clothing, miscellaneous) so that consumption in each category is always greater than zero. Families of utility functions yielding such “interior point” solutions to Equation (1) have been developed to exhibit a variety of useful properties, including behaviorally interpretable parameters and analytically tractable demand functions. A typical assumption is that the contribution to the direct utility from different goods is additively separable. The relaxation of this assumption and the assumption of a single linear constraint will be addressed in later sections.

Additional restrictions on the form of the utility function $U(\mathbf{x})$ in Equation (1) can be chosen that will yield either a single discrete-continuous (SDC) model or a multiple discrete-continuous (MDC) model. The SDC case with an outside good assumes that the inside goods are perfect substitutes, which ensures that no more than one inside good is chosen. The MDC case accommodates imperfect substitution among all goods, thus allowing for the possibility of choosing any combination of multiple alternatives. A

linear utility form with respect to consumption characterizes the perfect substitutes (or SDC) case. An example framework is due to Hanemann (1984):

$$U(\mathbf{x}) = U^*(\sum_{k=2}^{K} \psi_k x_k, x_1), \tag{2}$$

where U^* is a bivariate utility function, ψ_k (k = 2, …, K) represents a quality index (or baseline preference) specific to each inside good, and (in this case) the first good is the (numeraire) outside good. The form of Equation (2) ensures that, in addition to the numeraire good, no more than one inside good (k = 2, 3, …, K) is consumed. Hanemann (1984) refers to this as an "extreme corner solution".[1] Examples of MDC frameworks will be discussed later.

Two approaches have been used to derive demand functions for the consumption quantities for the utility maximization problem in Equation (1). The first approach, due to Hanemann (1978) and Wales and Woodland (1983), takes a direct approach to solving the constrained utility maximization problem in Equation (1) via standard application of the Kuhn-Tucker (KT) first-order necessary conditions of optimality.[2] Considering the utility function $U(\mathbf{x})$ to be random over the population leads to stochastic KT conditions, which form the basis for deriving probabilities for consumption patterns (including corner solutions). This approach is called the KT approach due to the central role played by the KT conditions.

A second approach due to Lee and Pitt (1986) solves the maximization problem in Equation (1) by using "virtual prices" (a method that is dual to the KT approach), which allows the analysis to start with the specification of a conditional indirect utility function. Subsequently, the implied Marshallian demand functions are obtained via Roy's identity (Roy 1947). Hanemann (1984) used this approach to derive a variety of SDC model

forms consistent with Equation (2). For SDC cases this approach can be much simpler to implement than the KT approach, both for model estimation and welfare analysis. Chiang (1991) and Chintagunta (1993) extend Hanemann's SDC formulation to include the possibility of no inside goods being selected by introducing a "reservation price". In their approach, an inside good is selected only if the quality adjusted price of at least one of the inside goods is below the reservation price. See Dubin and McFadden (1984) for another, slightly different, way of employing the (conditional) indirect utility approach for SDC choice analysis.

The vast majority of applications have involved single discrete or SDC choices. These use the indirect utility approach, due to its advantages in these cases. However, recent interest in MDC problems has brought renewed attention to the KT approach. The use of direct utility functions has some advantages: the relationship of the utility function to behavioral theory is more transparent, offering more interpretable parameters and better insights into identification issues. This is true even for the SDC case. For example, Bunch (2009) shows that the indirect utility function used by Chintagunta (1993) is in fact from the linear expenditure system, so the direct utility function is known. Applying the KT approach yields the correct analytical expression for the reservation price in terms of parameters from the direct utility function, which has a clear behavioral interpretation.

3. Recent advances with the Kuhn-Tucker approach to modeling MDC choices

Estimation, forecasting and welfare analysis for MDC models have seen limited use, primarily due to computational challenges associated with numerical evaluation of multidimensional integrals. Although both approaches discussed in Section 2 can have

advantages and disadvantages, there have been significant advances over the past decade using the KT approach.

3.1 Specification and estimation of Kuhn-Tucker demand systems

Kim et al. (2002) proposed a generalized variant of the translated Constant Elasticity of Substitution (CES) utility function $U(\mathbf{x}) = \sum_k \psi_k (x_k + \gamma_k)^{\alpha_k}$, where ψ_k is a quality index for good *k*, and α_k and γ_k are satiation and translation parameters, respectively. Specifically, an α_k parameter less than one allows for satiation through diminishing marginal utility with increasing consumption. Although sub-utilities for competing goods remain additive, satiation effects allow for consumption of multiple goods, i.e., goods act as imperfect substitutes. Positive γ_k parameters translate the utility function so that the indifference curves strike the consumption axes at an angle (rather than being asymptotic), and thereby allow corner solutions. Kim et al. introduce stochasticity by specifying the ψ_k terms as exponentials of normally distributed random terms, and obtain estimates using Markov Chain Monte Carlo. The model likelihood includes a multivariate normal integral, which they evaluate using the Geweke-Hajivassiliou-Keane (GHK) simulator. So, although the approach works for problems with a small number of choice alternatives, it becomes computationally intensive when the number of choice alternatives gets large.

A stream of research in the environmental economics field (see Phaneuf et al. 2000 and von Haefen and Phaneuf 2005) used the KT approach in the context of recreational demand analysis. These studies use variants of the linear expenditure system (LES) as proposed by Hanemann (1978), and also the translated CES utility

functions. They use multiplicative log-extreme value (log-EV) errors for the random utility specification to derive closed form probability expressions and log-likelihoods. However, they assume that the utility contribution due to the outside good is deterministic.

Bhat (2005) proposed a simple and parsimonious econometric approach to estimate KT demand systems by introducing a multiplicative log-extreme value (log-EV) specification in Kim et al.'s CES utility function. Unlike previous studies in environmental economics, he did not impose a deterministic utility function for the outside good. Bhat's model, denoted the multiple discrete-continuous extreme value (MDCEV) model, has several appealing properties. First, the model is analytically tractable with very simple probability expressions, and is estimable even in situations with a large number of choice alternatives. Second, when each (and every) consumer chooses one and only one alternative, the MDCEV model collapses to the well known single discrete choice multinomial logit (MNL) model.

More recently, Bhat (2008) introduced a Box-Cox transformation of the translated CES additive utility function:

$$U(\mathbf{x}) = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k} \psi_k \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\} \tag{3}$$

where ψ_k, γ_k and α_k are parameters associated with good *k*. Equation (3) is a valid utility function if $\psi_k > 0$ and $\alpha_k \leq 1$ for all *k*. The γ_k parameters, when positive, allow corner solutions similar to those in Kim et al. (2002). This utility form is particularly attractive for several reasons. First, it subsumes the translated CES and LES functions used in previous studies as special cases. Second, the structural parameters have a clearer behavioral interpretation than other utility functions in the literature. For example, the role of ψ_k can be inferred from the following marginal utility expression with respect

to good *k*: $\partial U(\boldsymbol{x})/\partial x_k = \psi_k \left(\left(x_k / \gamma_k \right) + 1 \right)^{\alpha_k - 1}$. The value ψ_k represents the *baseline marginal utility* (or marginal utility at zero consumption) for good *k*. Third, the utility form clarifies the inter-relationships among the structural parameters that relate to theoretical and empirical identification issues (Bhat 2008). Fourth, the effect of choice alternative quality attributes (z_k) can be incorporated in the baseline utility ψ_k (as $\psi_k = \exp(\beta' z_k)$) while maintaining the property of weak complementarity (Mäler 1974), i.e., the consumer receives no utility from a good's attributes if s/he does not consume it. Fifth, stochasticity (ε_k) can be incorporated into ψ_k in a multiplicative form as: $\psi_k = \exp(\beta' z_k) \times \exp(\varepsilon_k)$. As indicated earlier, assuming an IID extreme value (EV) distribution for $\varepsilon_k \; \forall k$ (i.e., a multiplicative log-EV form) leads to simple and elegant analytical probability expressions from the stochastic KT conditions. As a result, estimation of the parameters becomes very easy.

Because of these advances, several empirical applications have appeared in the recent literature using the KT approach to model MDC choices. These applications cover a wide range of empirical contexts, including individuals' time-use analysis, household expenditure patterns, household vehicle ownership and usage, household energy consumption, recreational demand choices, and valuation of a variety of environmental goods (e.g., fish stock, air quality, water quality). It is now possible to estimate KT demand systems with more than just a few choice alternatives (see Van Nostrand et al. 2012 for a model with 211 choice alternatives).

3.2 Flexible specifications

Building on the above advances, recent research has focused on extending the basic RUM specification for Equation (1) in various ways, such as: (a) specification of heterogeneity in consumer preferences, and (b) allowing for flexible stochastic specifications.

Advances in computing power, and developments in simulation and Bayesian methods (von Haefen and Phaneuf 2005), have been exploited to estimate richer specifications of unobserved heterogeneity in consumer preferences. For example, the closed form probability expression for the MDCEV model allows it to be used as a kernel for simulation estimation of random coefficient models that use continuous multivariate probability distributions to represent preference heterogeneity, analogous to many applications of the mixed multinomial logit model (McFadden and Train 2000) for single discrete choice applications. Similarly, discrete mixing distributions can be used to implement latent class specifications for market segmentation in KT models (Kuriyama et al. 2010). Within the context of heterogeneity, the issue of latent consideration choice sets has been addressed by von Haefen (2008) and Castro et al. (2012). Both papers adopt a two-stage decision-making approach, where the first stage includes a probabilistic generation of the subset of choice alternatives considered by the decision-makers, and the second stage applies the KT framework using only the subset of choices generated in the first stage. In addition to addressing consideration set formation, this approach supports model specifications that are more flexible in their ability to “decouple” the way in which explanatory variables affect the discrete versus continuous components of decision making.[3]

Another recent research direction has been to relax the assumption of independent and identically distributed (IID) random terms in the utility functions. The IID assumption, although commonly used for convenience, may not be justified in many empirical contexts, as it can potentially result in: (1) neglect of unobserved random effects that are shared in common across sets of choice alternatives, (2) inability to accommodate certain types of flexible substitution patterns, and (3) biased parameter estimates, poor model fit, and distorted policy implications. One way to relax this assumption is to extend the random coefficients approach to include error components that are shared across varying sets of choice alternatives. This is a way to implement correlation among the error terms, thus relaxing the typical IID assumption. More recently, Pinjari and Bhat (2010) and Pinjari (2011) employ multivariate extreme value (MEV) specifications that directly relax the IID assumption while retaining closed form probability expressions (also see Phaneuf et al. 2000).

3.3 Prediction and welfare analysis

Once model parameters are estimated, applying the model to policy analysis or decision making typically requires prediction exercises or welfare analyses under alternative scenarios. The usual approach is to use the original data set of consumers as the framework for performing the required computations. For KT-based MDC models, “simulating” the probability distribution of decisions for each consumer requires solving the problem in equation (1) under the assumption of random utility maximization. In the presence of corner solutions (i.e., multiple discreteness), there is no straightforward analytic solution to this problem. The typical approach is to adopt a constrained non-linear optimization procedure at each of several simulated values drawn from the

distribution of unobserved heterogeneity (or stochasticity due to the RUM formulation). The constrained optimization procedure itself has been based on either enumerative or iterative techniques. The enumerative technique (used by Phaneuf et al. 2000) involves an enumeration of all possible sets of alternatives that the consumer can potentially choose. This brute-force method becomes computationally impractical as the number of choice alternatives increases. Thus, von Haefen et al. (2004) proposed a numerical bisection algorithm based on the insight that, with additively separable utility functions, the optimal consumptions of all goods can be derived if the optimal consumption of the outside good is known. Specifically, conditional on unobserved heterogeneity, they iteratively solve for the optimal consumption of the outside good (and that of other goods) using a bisection procedure.

In a recent paper, Pinjari and Bhat (2011) provide an analysis of the KT optimality conditions for Bhat's MDCEV model that sheds new light on the properties of KT models with additive utility functions. Specifically, they show that, for the MDCEV model, the price-normalized baseline marginal utility (i.e., ψ_k / p_k) of a chosen alternative must be greater than the price-normalized baseline marginal utility of any alternative that is not chosen. Further, they discuss a fundamental property of several KT demand model systems with additively separable utility functions and a single linear binding constraint. Specifically, choice alternatives can always be arranged in descending order of a specific measure that depends on the functional form of the utility function (e.g., the price-normalized baseline marginal utility in case of the MDCEV model). Consequently, when the alternatives have been so arranged, and the number of chosen alternatives (*M*) is known, it is a trivial task to identify the chosen alternatives as the first *M* alternatives in the arrangement. Based on this insight, Pinjari and Bhat (2011) propose

an efficient, non-iterative, prediction algorithm for a specific form of the utility in Equation (3) with α_k parameters equal across all choice alternatives (i.e., the LES form). Using the same insight, they formulate prediction algorithms for other additive utility functions (i.e., when α_k parameters differ across choice alternatives).

The above discussion is primarily oriented toward using KT-based MDC models for prediction, but does not extend the discussion to include welfare analysis. For a discussion of how such prediction algorithms can be used for welfare analysis, see von Haefen and Phaneuf (2005).

4. Future directions

There has recently been increased recognition of the need to extend MDC models based on Equation (1) in the following directions:

(1) More flexible functional forms for the utility specification,

(2) More flexible stochastic specifications for the utility functions,

(3) Greater flexibility in the specification of constraints faced by the consumer.

Each of these is discussed in turn in the next three sections.

4.1 Flexible, non-additive utility forms

Most KT models in the literature assume that the direct utility contribution from consumption of different alternatives is additively separable. Mathematically, this assumption implies that: $U(x_1,...,x_K) = U_1(x_1) + ... + U_K(x_K)$, and greatly simplifies the task of model estimation and welfare analysis. However, this assumption imposes strong restrictions on preference structures and consumption patterns. Specifically, the

marginal utility of one alternative is independent of the consumption level of another alternative. For an increasing and quasi-concave utility function, this assumption implies that goods can be neither inferior nor complementary; they can only be substitutes. For example, one cannot model a situation where the increased consumption of one good (e.g., an automobile) may increase the utility of consuming another good (e.g., gasoline). Although some flexibility in substitution patterns for additive utility forms can be achieved by correlating the stochastic utility components of different goods, this flexibility is quite limited compared to what could be achieved by directly modeling preference interactions through the explicit functional form of the utility function. To overcome the restrictions identified above, it is critical to develop tractable estimation methods with flexible, non-additively separable utility functions.

There have been a handful of recent efforts in this direction. For example, building on Bhat's additively separable linear Box-Cox utility form, Vasquez Lavin and Hanemann (2008) presented a general utility form with interaction terms between sub-utilities, as below:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \psi_k \frac{\gamma_k}{\alpha_k} \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\} + \frac{1}{2} \sum_{k=1}^{K} \sum_{m=1}^{K} \left\{ \theta_{km} \frac{\gamma_k}{\alpha_k} \left[\left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right] \frac{\gamma_m}{\alpha_m} \left[\left(\frac{x_m}{\gamma_m} + 1 \right)^{\alpha_m} - 1 \right] \right\} \tag{4}$$

In the above expression, the second term induces interactions between pairs of goods (*m,k*) and includes quadratic terms (when *m* = *k*). These interaction terms allow the marginal utility of a good (*k*) to depend on the consumption of other goods (*m*). Specifically, a positive (negative) value for θ_{mk} implies that *m* and *k* are complements (substitutes). However, the quadratic nature of the utility form does not maintain global consistency (over all consumption bundles) of the strictly increasing and quasi-concave property. Specifically, for certain parameter values and consumption patterns, the utility

accrued can *decrease* with increasing consumption, or the marginal utility can *increase* with increasing consumption, which is theoretically inconsistent. Bhat and Pinjari (2010) show how a simple normalization by setting $\theta_{mk} = 0$ when *m* = *k* in Equation (4) can resolve the issues of theoretical (in)consistency and parameter (un)identification. Other efforts to accommodate complementarity in consumption include Lee et al. (2010) who propose simpler interaction terms using log(quantities), and Gentzkow (2007) who accommodates interactions in indirect utility functions.

Despite the above efforts, there are still unresolved conceptual and methodological issues pertaining to: (1) the form of non-additive utility functions, (2) the specification of stochasticity in non-additive utility functions, (3) estimation of parameters with increasing numbers of choice alternatives, and (4) interpretation of the resulting dependency patterns in consumption. Resolving these issues will be a big step forward in enhancing the behavioral realism of KT-based RUM MDC models. Further, within the context of non-additively separable preferences, it is important to recognize asymmetric dependencies in consumption. For example, the purchase of a new car may lead to increased gasoline consumption, but not the other way round.

4.2 Flexible stochastic specifications

The above discussion was in the context of the form of the utility function. But there is potential for improving the stochastic specification as well. For example, most studies assume IID log-extreme value (log-EV) stochasticity in the utility function. Recent advances on relaxing the IID assumption, specifically via employing MEV distributions, have been discussed in Section 3.2. Although we are now able to estimate KT-based RUM MDC models with general MEV distributions, no clear understanding exists on how

different stochastic specifications and utility functional forms influence the properties of KT models. Examining the substitution patterns implied by the different stochastic assumptions in KT-based MDC models is a useful avenue for additional research.

The choice of extreme value (either EV or MEV) stochastic specification is driven by convenience (of analytical tractability) rather than theory. It is well recognized that an assumption of multivariate normally (MVN) distributed stochastic error terms leads to complex multivariate integrals, which was one reason why KT approach did not gain traction for empirical analysis until recently. As discussed earlier, attempts have been made to address this issue by using simulation methods (such as the GHK simulator) and Bayesian estimation methods. However, the GHK and other such simulators become computationally impractical as the dimensionality of integration increases with the number of alternatives. Bayesian estimation methods can also be computationally intensive and are saddled with convergence-determination issues. Thus, no study has estimated KT demand systems with MVN distributions for more than a small number of alternatives. In the recent past, there has been some evidence that using analytical approximations (rather than simulation) for evaluating the MVN cumulative distribution function can reduce the computational requirements for single discrete choice models (e.g., the multinomial probit model; see Bhat and Sidharthan 2011). It may be that such analytical approximation methods can help in the estimation of MDC models with MVN errors. An appealing feature of MVN errors is the possibility of negative correlations among the utilities of different alternatives (as opposed to MEV errors, which allow only positive dependency). This can potentially be exploited to capture situations where the choice of one alternative may reduce (if not preclude) the likelihood of choosing another,

where the pattern of substitution is fundamentally different from the substitution due to satiation effects.

Finally, as noted previously, several studies in the literature impose a deterministic utility specification for the outside numeraire good (i.e., an essential Hicksian composite good) and use a multiplicative log-EV stochastic specification for the other (inside) goods. Although this can lead to analytically tractable likelihood expressions, it is rather arbitrary to assume away stochasticity on the numeraire good. Moreover, this approach can break down in cases without an outside good. At the same time, it is possible that such specifications will reduce the complexity of the estimation problem when adopting flexible utility functions (such as non-additive utility forms) in the presence of an outside good. Given the potential computational advantages on the one hand versus the arbitrariness of the deterministic assumption (on the outside good) on the other, it may be useful to conduct empirical studies assessing the impact of this assumption on model fit, predictions, and welfare measures.

4.3 Multiple constraints

Most MDC modeling applications to date consider only a single linear binding constraint (e.g., the linear constraint in Equation (1)). This stems from an implicit assumption that only a single resource is needed to consume goods. However, in numerous empirical contexts, multiple types of resources, such as time, money and space, need to be expended to acquire and consume goods.

While the role of multiple constraints has been long recognized in microeconomic theory (see Becker 1965), the typical approach to accommodating multiple constraints has been to combine them into a single effective constraint. For example, the time

constraint has been collapsed into the money constraint by using a monetary value of time. In many situations, however, it is important to consider the different constraints in their own right, because resources may not always be freely exchangeable with one other. To address this issue, a handful of recent studies (Satomura et al. 2011, Castro et al. 2012, Pinjari and Sivaraman 2012) have provided model formulations that accommodate multiple linear constraints with additive utility functional forms. Satomura et al. (2011) provided a formulation to account for the roles of money and space constraints in consumers' decisions on soft drink purchases. Castro et al. (2012) provide a general treatment of the issue by providing formulations for different scenarios such as complete demand systems (i.e., a case without the need of a Hicksian composite good), and incomplete demand systems (a case with the Hicksian composite good). Pinjari and Sivaraman (2012) provide a time- and money-constrained formulation in the context of households' annual vacation travel choices with a large number of destination choice alternatives.

4.4 Beyond simple, linear constraints

The above discussion suggests that we have just begun to move toward models with multiple constraints. However, another concern is the assumption of linearity. Simple, linear constraints may not accurately represent many of the situations consumers face in reality. There are several reasons why the linearity assumption may not hold. First, the usual linear expenditure constraint assumes a constant price per unit of consumption. In many situations, however, prices vary with the amount of consumption, i.e. budget constraints are actually non-linear. A classic example is the block pricing typically used in energy markets (e.g., electricity pricing).[4] While the issue has long been recognized in

the classical econometric literature on estimating demand functions, it has yet to be given due consideration in MDC choice studies. Second, linear constraints do not accommodate fixed costs (or setup costs). For example, travel cost to a vacation destination is a *fixed* cost, unlike the lodging costs at the destination which can be treated as *variable* with a constant price per night.

Using the KT approach to solve the consumer's direct utility maximization problem can be very difficult when the constraints are nonlinear. In a recent study, Parizat and Shachar (2010) employ an enumeration approach to solve a direct utility maximization problem in the context of individuals' weekly leisure time allocation with fixed costs (e.g., ticket costs of going to a movie, the price of a meal). They report that rather large computation times are required to estimate the parameters for their 12-alternative case. An alternative approach to incorporating nonlinear constraints may be to use indirect utility functions and virtual prices: Lee and Pitt (1987) develop such an approach and apply it to a problem with block pricing. Further studies exploring this approach may enhance our ability to incorporate block prices. Finally, another way to address pricing problems is to explicitly model decision making for both supply and demand, so that prices are determined endogenously as part of the estimation process (see Berry et al. 1995).

4.5 Prediction and welfare analysis with flexible model structures

Section 3.3 discussed recent advances for performing prediction and welfare analysis with KT models; however, these approaches are for models based on additively separable utilities with a linear constraint. As the field moves forward with the specification and estimation of more flexible MDC models, it is important to develop

prediction methods for these models as well. The procedures proposed by von Haefen et al. (2004) and Pinjari and Bhat (2011) based on Kuhn-Tucker optimality conditions can potentially be extended to the case of multiple linear binding constraints as well, although with additional layers of computational effort (as many as the number of constraints). However, these procedures fall apart in situations with non-additive utility functions, as they critically rely on the additive utility assumption. And, as in the case of model estimation discussed in Section 4.4, non-linear constraints can make it difficult to apply KT conditions as an approach to solving the utility maximization problem. Resolving each of these issues would be a welcome outcome of future research.

Although directions for future research discussed in this section are primarily concerned with more flexible utility and constraint functions, there are still potentially important contributions to be made within the context of additive utility functions with a simple linear constraint (as in Equation 1). While we are now able to exploit the KT conditions for obtaining the conditional predictions (given specific values of the random terms), we have not been able to characterize the unconditional distributions of the demand functions. In the presence of corner solutions, it is difficult to arrive at closed form expressions for the demand functions arising from Equation (1). Perhaps this is why we are not aware of successful attempts to develop analytical expressions for price elasticities and sensitivities to explanatory variables. As in other applications of these models, evaluating these expressions requires the simulation of stochastic terms. In some cases, such as models based on MEV distributions, the model estimation advantages of the closed-form probability expressions do not carry over to prediction. In fact, simply simulating the stochastic terms from an MEV distribution is a challenging task that could benefit from future research efforts.

5. Summary

There has been an increasing recognition of the “multiple discrete-continuous (MDC)” nature of many important consumer choices. Over the past decade, the field has witnessed exciting developments in modeling MDC choices, especially with the advancement of the Kuhn-Tucker (KT) approach to modeling consumer behavior based on random utility maximization (RUM). Notable developments include:

1. Clever specifications with distributional assumptions that lead to closed-form probability expressions enabling easy estimation of the structural parameters (e.g., the MDCEV model),
2. Application of the KT approach to model MDC choices in a variety of empirical contexts,
3. Formulation of computationally efficient prediction/welfare analysis methods with KT models,
4. Extension of the basic RUM specification in Equation (1) to accommodate richer patterns of heterogeneity in consumer preferences and to allow flexibility in distributional assumptions for the stochastic error term. Most of these extensions have been “econometrically” oriented, akin to the extensions of the multinomial logit model in the traditional discrete choice analysis literature.

In the recent past, there has been an increasing recognition of the need to extend the basic formulation of the consumer’s utility maximization problem in Equation (1) in the following directions:

1. More flexible functional forms for the utility specification, such as non-additive utility forms,
2. More flexible stochastic specifications for the utility functions, such as MVN distributions,
3. Greater flexibility in the specification of constraints faced by the consumer, including multiple constraints and non-linear constraints.

Given the pace of recent developments, we optimistically look forward to seeing model formulations, estimation methods, and prediction/welfare analysis procedures for a general framework with non-additive utility forms, flexible stochastic distributional assumptions, and general forms of constraints. Other more general directions for future research include the extension of the KT approach to model consumer decisions over long time-frames with inter-temporal dependency in consumption (e.g., using dynamic programming), and the incorporation of findings from experimental data related to consumer behavior, beliefs and attitudes into structural KT demand models.

Acknowledgements

The authors thank the workshop participants for insightful and lively discussions on the topic. David Bunch and Abdul Pinjari made initial presentations to provide a basis for the workshop discussions.

References

Becker, G.S. (1965), ‘A theory of the allocation of time’, *The Economic Journal*, **75** (299), 493-517.

Berry, S., J. Levinsohn and A. Pakes (1995), ‘Automobile prices in market equilibrium’, *Econometrica*, **60** (4), 841-890.

Bhat, C.R. (2005), ‘A multiple discrete-continuous extreme value model: formulation and application to discretionary time-use decisions’, *Transportation Research Part B*, **39** (8), 679-707.

Bhat, C.R. (2008), ‘The multiple discrete-continuous extreme value (MDCEV) model: role of utility function parameters, identification considerations, and model extensions*’, Transportation Research Part B*, **42** (3), 274-303.

Bhat, C.R. and A.R. Pinjari (2010), ‘The generalized multiple discrete-continuous extreme value (GMDCEV) model: allowing for non-additively separable and flexible utility forms’, working paper, Department of Civil, Architectural and Environmental Engineering, The University of Texas at Austin.

Bhat, C.R. and R. Sidharthan (2011), ‘A simulation evaluation of the maximum approximate composite marginal likelihood (MACML) estimator for mixed multinomial probit models’, *Transportation Research Part B*, **45** (7), 940-953.

Bunch, D.S. (2009), ‘Theory-based functional forms for analysis of disaggregated scanner panel data’, working paper, Graduate School of Management, University of California-Davis.

Castro, M., C.R. Bhat, R.M. Pendyala and S.R. Jara-Diaz (2012), 'Accommodating multiple constraints in the multiple discrete-continuous extreme value (MDCEV) choice model', *Transportation Research Part B*, **46** (6), 729-743.

Chiang J. (1991), 'The simultaneous approach to the whether, what, and how much to buy questions' *Marketing Science*, **10** (4), 297-315.

Chintagunta, P. (1993), 'Investigating purchase incidence, brand choice and purchase quantity decisions of households' *Marketing Science*, **12** (2), 184-208.

Deaton, A. and J. Muellbauer (1980), *Economics and Consumer Behavior*, Cambridge, UK: Cambridge University Press.

Dubé, J-P. (2004), 'Multiple discreteness and product differentiation: demand for carbonated soft drinks', *Marketing Science*, **23** (1), 66-81.

Dubin, J. and D. McFadden (1984), 'An econometric analysis of electricity appliance holdings and consumption', *Econometrica*, **52** (2), 345-362.

Gentzkow, M. (2007), 'Valuing new goods in a model with complementarity: online newspapers', *American Economic Review*, **97** (3), 713-744.

Hanemann, W.M. (1978), ‘A methodological and empirical study of the recreation benefits from water quality improvement’, Ph.D. dissertation, Department of Economics, Harvard University.

Hanemann, W.M. (1984), ‘Discrete/continuous models of consumer demand’, *Econometrica*, **52** (3), 541-561.

Hendel, I. (1999), ‘Estimating multiple-discrete choice models: an application to computerization returns’, *Review of Economic Studies*, **66** (2), 423-446.

Kim, J., G.M. Allenby and P.E. Rossi (2002), ‘Modeling consumer demand for variety’, *Marketing Science*, **21** (3), 229-250.

Kuriyama, K., W.M. Hanemann and J.R. Hilger (2010), ‘A latent segmentation approach to a Kuhn-Tucker model: an application to recreation demand’, *Journal of Environmental Economics and Management*, **60** (3), 209-220.

Lee, L.F. and M.M. Pitt (1986), ‘Microeconometric demand systems with binding nonnegativity constraints: the dual approach’, *Econometrica*, **54** (5), 1237-1242.

Lee, L.F. and M.M. Pitt (1987), ‘Microeconometric models of rationing, imperfect markets, and non-negativity constraints’, *Journal of Econometrics,* **36** (1-2), 89-110.

Lee, S., J. Kim and G. Allenby (2010), 'A direct utility model for asymmetric complements', working paper, Department of Marketing and Logistics, Ohio State University.

Mäler, K-G. (1974), *Environmental Economics: A Theoretical Inquiry*. Baltimore, MD: The Johns Hopkins University Press for Resources for the Future.

McFadden, D. and K. Train (2000), 'Mixed MNL models for discrete response' *Journal of Applied Econometrics*, **15** (5), 447–470.

Parizat, S. and R. Shachar (2010), 'When Pavarotti meets Harry Potter at the Super Bowl', working paper, Tel Aviv University.

Pinjari, A.R. (2011), 'Generalized extreme value (GEV)-based error structures for multiple discrete-continuous choice models', *Transportation Research Part B*, **45** (3), 474-489.

Pinjari, A.R. and C.R. Bhat (2010), 'A multiple discrete-continuous nested extreme value (MDCNEV) model: formulation and application to non-worker activity time-use and timing behavior on weekdays', *Transportation Research Part B*, **44** (4), 562-583.

Pinjari, A.R. and C.R. Bhat (2011), 'Computationally efficient forecasting procedures for Kuhn-Tucker consumer demand model systems: application to residential energy consumption

analysis', working paper, Department of Civil and Environmental Engineering, University of South Florida.

Pinjari, A.R. and V. Sivaraman (2012), 'A time- and money-constrained model of household vacation travel behavior: formulation and application of a Kuhn-Tucker demand model system', Proceedings of the 13th Conference of the International Association for Travel Behaviour Research (IATBR), Toronto, Canada, July 2012.

Phaneuf, D.J., C.L. Kling and J.A. Herriges (2000), 'Estimation and welfare calculations in a generalized corner solution model with an application to recreation demand', *The Review of Economics and Statistic*, **82** (1), 83-92.

Roy, R. (1947), 'La distribution du revenu entre les divers biens', *Econometrica*, **15** (3), 205-225.

Satomura, S., J. Kim and G. Allenby (2011), 'Multiple constraint choice models with corner and interior solutions', *Marketing Science*, **30** (3), 481-490.

Vasquez Lavin, F.A. and W.M. Hanemann (2008), 'Functional forms in discrete/continuous choice models with general corner solution', CUDARE Working Paper Series, no 1078, Department of Agricultural and Resource Economics and Policy, University of California at Berkeley.

Van Nostrand, C., V. Sivaraman and A.R. Pinjari (2012), 'Analysis of long-distance, vacation travel demand in the United States: a multiple discrete-continuous choice framework', *Transportation*, forthcoming.

von Haefen, R.H. (2008), 'Latent consideration sets and continuous demand system models', *Environmental and Resource Economics*, **41** (3), 363-379.

von Haefen, R.H. and D.J. Phaneuf (2005), 'Kuhn-Tucker demand system approaches to nonmarket valuation', in R. Scarpa and A. Alberini (eds.), *Applications of Simulation Methods in Environmental and Resource Economics*, Dordrecht, The Netherlands: Springer, pp. 135-158.

von Haefen, R.H., D.J. Phaneuf and G.R. Parsons (2004), 'Estimation and welfare analysis with large demand systems', *Journal of Business and Economic Statistics,* **22** (2), 194-205.

Wales, T.J. and A.D. Woodland (1983), 'Estimation of consumer demand systems with binding non-negativity constraints', *Journal of Econometrics*, **21** (3), 263-285.

---

[1] Under appropriate assumptions, extreme corner solution models from this framework can also accommodate the case where none of the inside goods is consumed.

[2] The applied mathematics literature now refers to these as Karush-Kuhn-Tucker (KKT) conditions to appropriately assign historical credit to Karush who developed the conditions well before Kuhn and Tucker. We retain the KT nomenclature for consistency with much of the relevant microeconomics literature cited here.

[3] There seems to be the (incorrect) notion that the KT approach requires a tight nexus between the discrete and continuous components of choice. Although many KT-based models have this property, the two-stage modeling approach helps to ease the nexus. Another, simpler approach

to doing so is to enhance the empirical specification of the satiation parameter functions (γ_k and α_k). Since these parameters govern the rate of satiation (hence the decision on how much to consume), a carefully specified satiation function that incorporates alternative sets of attributes and demographic characteristics can help distinguish the influence of explanatory variables on the continuous and discrete components of choice.

[4] In this case the price constraint is not just nonlinear, it is also non-smooth, which can complicate model estimation.